The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

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Topic Generator - Solution Set
Solutions

1. The average (mean) of the numbers 6, 8, 9, 11, and 16 is
   (A) 8 (B) 9 (C) 10 (D) 11 (E) 7

Source: 2005 Gauss Grade 8 #4

Primary Topics: Data Analysis

Secondary Topics: Averages

Answer: C

Solution:

The mean of these five numbers is $\frac{6+8+9+11+16}{5} = \frac{50}{5} = 10$.

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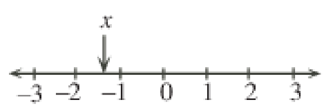
2. Which of the following is the best estimate for the value of x shown on the number line?



   (A) 1.3 (B) -1.3 (C) -2.7 (D) 0.7 (E) -0.7

Source: 2007 Pascal Grade 9 #2

Primary Topics: Number Sense

Secondary Topics: Estimation

Answer: B

Solution:

Since x is less than -1 and greater than -2, then the best estimate of the given choices is -1.3.

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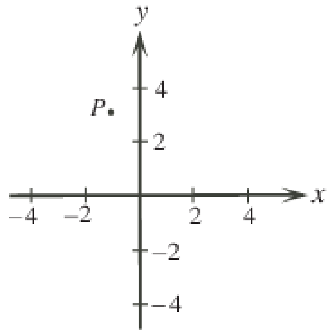
3. In the diagram, the coordinates of point P could be



(A) $(1,3)$ (B) $(1,-3)$ (C) $(-3,1)$ (D) $(3,-1)$ (E) $(-1,3)$

Source: 2009 Gauss Grade 7 #5

Primary Topics: Geometry and Measurement

Secondary Topics: Graphs | Estimation

Answer: E

Solution:

The x-coordinate of point P lies between -2 and 0. The y-coordinate lies between 2 and 4.
Of the possible choices, $(-1,3)$ is the only point that satisfies both of these conditions.

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4. The mean (average) of 5 consecutive integers is 9. What is the smallest of these 5 integers?
(A) 4 (B) 5 (C) 6 (D) 7 (E) 8

Source: 2010 Cayley Grade 10 #7

Primary Topics: Number Sense

Secondary Topics: Averages

Answer: D

Solution:

Solution 1
Since the mean of five consecutive integers is 9, then the middle of these five integers is 9.
Therefore, the integers are $7, 8, 9, 10, 11$, and so the smallest of the five integers is 7.

Solution 2
Suppose that x is the smallest of the five consecutive integers.
Then the integers are x, $x+1$, $x+2$, $x+3$, and $x+4$.
The mean of these integers is $\frac{x+(x+1)+(x+2)+(x+3)+(x+4)}{5}=\frac{5x+10}{5}=x+2$.
Since the mean is 9, then $x+2=9$ or $x=7$.
Thus, the smallest of the five integers is 7.

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5. Which of the following numbers is a multiple of 9?
(A) 50 (B) 40 (C) 35 (D) 45 (E) 55

Source: 2013 Gauss Grade 7 #2

Primary Topics: Number Sense

Secondary Topics: Divisibility

Answer: D

Solution:

Solution 1
A number is a multiple of 9 if it is the result of multiplying 9 by an integer.
Of the answers given, only 45 results from multiplying 9 by an integer, since $45 = 9 \times 5$.
Solution 2
A number is a multiple of 9 if the result after dividing it by 9 is an integer.
Of the answers given, only 45 results in an integer after dividing by 9, since $45 \div 9 = 5$.

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6. The value of $2^4 - 2^3$ is
(A) 0^1 (B) 2^1 (C) 2^2 (D) 2^3 (E) 1^1

Source: 2014 Pascal Grade 9 #9

Primary Topics: Number Sense

Secondary Topics: Operations | Exponents

Answer: D

Solution:

We note that $2^2 = 2 \times 2 = 4$, $2^3 = 2^2 \times 2 = 4 \times 2 = 8$, and $2^4 = 2^2 \times 2^2 = 4 \times 4 = 16$.
Therefore, $2^4 - 2^3 = 16 - 8 = 8 = 2^3$.

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7. Which of the following numbers is greater than 0.7?
(A) 0.07 (B) -0.41 (C) 0.8 (D) 0.35 (E) -0.9

Source: 2015 Pascal Grade 9 #4

Primary Topics: Number Sense

Secondary Topics: Estimation

Answer: C

Solution:

Each of 0.07, -0.41, 0.35, and -0.9 is less than 0.7 (that is, each is to the left of 0.7 on the number line).
The number 0.8 is greater than 0.7.

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8. If $x = 3$, $y = 2x$, and $z = 3y$, then the average of x, y and z is
   (A) 6 (B) 7 (C) 8 (D) 9 (E) 10

Source: 2016 Pascal Grade 9 #8

Primary Topics: Algebra and Equations

Secondary Topics: Averages

Answer: D

Solution:

Since $x = 3$ and $y = 2x$, then $y = 2 \times 3 = 6$.
Since $y = 6$ and $z = 3y$, then $z = 3 \times 6 = 18$.
Therefore, the average of x, y and z is $\frac{x+y+z}{3} = \frac{3+6+18}{3} = 9$.

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9. Alexis took a total of $243\,000$ steps during the 30 days in the month of April. What was her mean (average) number of steps per day in April?
   (A) 7900 (B) 8100 (C) 8000 (D) 7100 (E) 8200

Source: 2020 Gauss Grade 7 #5

Primary Topics: Number Sense

Secondary Topics: Averages

Answer: B

Solution:

In April, Alexis averaged $243\,000 \div 30 = 8100$ steps per day.

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10. Elena earns \$13.25 per hour working at a store. How much does Elena earn in 4 hours?
    (A) \$54.00 (B) \$56.25 (C) \$52.25 (D) \$51.00 (E) \$53.00

Source: 2021 Pascal Grade 9 #3

Primary Topics: Number Sense

Secondary Topics: Operations

Answer: E

Solution:

Elena works for 4 hours and earns \$13.25 per hour.

This means that she earns a total of $4 \times \$13.25 = \$53.00$.

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11. A student may pay \$1.50 for a single bus ticket or \$5.75 for a package of 5 tickets. If a student requires 40 tickets, how much does she save by buying all of the tickets in packages of 5 rather than buying 40 single tickets?
(A) \$54.25 (B) \$34.00 (C) \$14.00 (D) \$8.25 (E) \$4.25

Source: 2005 Gauss Grade 8 #16

Primary Topics: Number Sense | Algebra and Equations

Secondary Topics: Decimals | Operations

Answer: C

Solution:

If the student were to buy 40 individual tickets, this would cost $40 \times \$1.50 = \$60.00$.
If the student were to buy the tickets in packages of 5, she would need to buy $40 \div 5 = 8$ packages, and so this would cost $8 \times \$5.75 = \$46.00$.
Therefore, she would save $\$60.00 - \$46.00 = \$14.00$.

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12. Sally picks four consecutive positive integers. She divides each integer by four, and then adds the remainders together. The sum of the remainders is
(A) 6 (B) 1 (C) 2 (D) 3 (E) 4

Source: 2007 Gauss Grade 8 #15

Primary Topics: Number Sense

Secondary Topics: Divisibility

Answer: A

Solution:

Let us try the integers 5, 6, 7, 8.
When 5 is divided by 4, the quotient is 1 and the remainder is 1.
When 6 is divided by 4, the quotient is 1 and the remainder is 2.
When 7 is divided by 4, the quotient is 1 and the remainder is 3.
When 8 is divided by 4, the quotient is 2 and the remainder is 0.
The sum of these remainders is $1 + 2 + 3 + 0 = 6$.
(When any four consecutive integers are chosen, one will have a remainder 1, one a remainder of 2, one a remainder of 3 and one a remainder of 0 when divided by 4.)

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13. If x and y are two-digit positive integers with $xy = 555$, what is $x + y$?
(A) 52 (B) 116 (C) 66 (D) 555 (E) 45

Source: 2008 Cayley Grade 10 #15

Primary Topics: Algebra and Equations

Secondary Topics: Divisibility

Answer: A

Solution:

First, we find the prime factors of 555.
Since 555 ends with a 5, it is divisible by 5, with $555 = 5 \times 111$.
Since the sum of the digits of 111 is 3, then 111 is divisible by 3, with $111 = 3 \times 37$.
Therefore, $555 = 3 \times 5 \times 37$, and each of 3, 5 and 37 is a prime number.
The possible ways to write 555 as the product of two integers are 1×555, 3×185, 5×111, and 15×37. (In each of these products, two or more of the prime factors have been combined to give a composite divisor.)
The only pair where both members are two-digit positive integers is 37 and 15, so $x + y$ is $37 + 15 = 52$.

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14. Which of the following expressions is equal to 5?
(A) $(2 \times 3)^2$ (B) $3 + 2^2$ (C) $2^3 - 1$
(D) $3^2 - 2^2$ (E) $(3 + 2)^2$

Source: 2011 Gauss Grade 8 #11

Primary Topics: Number Sense

Secondary Topics: Operations | Exponents

Answer: D

Solution:

Evaluating each of the expressions,
(A): $(2 \times 3)^2 = 6^2 = 36$
(B): $3 + 2^2 = 3 + 4 = 7$
(C): $2^3 - 1 = 8 - 1 = 7$
(D): $3^2 - 2^2 = 9 - 4 = 5$
(E): $(3 + 2)^2 = 5^2 = 25$,
we see that only expression (D) is equal to 5.

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15. Which of the following numbers is closest to 1?
(A) $\frac{11}{10}$ (B) $\frac{111}{100}$ (C) 1.101 (D) $\frac{1111}{1000}$ (E) 1.011

Source: 2011 Pascal Grade 9 #11

Primary Topics: Number Sense

Secondary Topics: Decimals | Fractions/Ratios | Estimation

Answer: E

Solution:

When we convert each of the possible answers to a decimal, we obtain 1.1, 1.11, 1.101, 1.111, and 1.011.
Since the last of these is the only one greater than 1 and less than 1.1, it is closest to 1.

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16. Which of the following is *not* equal to $\frac{15}{4}$?
(A) 3.75 (B) $\frac{14+1}{3+1}$ (C) $\frac{3}{4}+3$ (D) $\frac{5}{4}\times\frac{3}{4}$ (E) $\frac{21}{4}-\frac{5}{4}-\frac{1}{4}$

Source: 2012 Gauss Grade 7 #12

Primary Topics: Number Sense

Secondary Topics: Fractions/Ratios | Operations

Answer: D

Solution:

Since $\frac{14+1}{3+1}=\frac{15}{4}$ and $\frac{21}{4}-\frac{5}{4}-\frac{1}{4}=\frac{16}{4}-\frac{1}{4}=\frac{15}{4}$, then answers (B) and (E) both simplify to $\frac{15}{4}$.
Written as a mixed fraction, $\frac{15}{4}$ is equal to $3\frac{3}{4}$.
Since $3.75=3\frac{3}{4}=3+\frac{3}{4}$, then answers (A) and (C) both simplify to $3\frac{3}{4}$ and thus are equivalent to $\frac{15}{4}$.
Simplifying answer (D), $\frac{5}{4}\times\frac{3}{4}=\frac{5\times3}{4\times4}=\frac{15}{16}$.
Thus, $\frac{5}{4}\times\frac{3}{4}$ is not equal to $\frac{15}{4}$.

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17. Integers greater than 1000 are created using the digits 2, 0, 1, 3 exactly once in each integer. What is the difference between the largest and the smallest integers that can be created in this way?

(A) 2187 (B) 2333 (C) 1980 (D) 3209 (E) 4233

Source: 2013 Cayley Grade 10 #14

Primary Topics: Number Sense

Secondary Topics: Optimization | Digits

Answer: A

Solution:

With a given set of four digits, the largest possible integer that can be formed puts the largest digit in the thousands place, the second largest digit in the hundreds place, the third largest digit in the tens place, and the smallest digit in the units place. This is because the largest digit can make the largest contribution in the place with the most value.
Thus, the largest integer that can be formed with the digits 2, 0, 1, 3 is 3210.
With a given set of digits, the smallest possible integer comes from listing the numbers in increasing order from the thousands place to the units place.
Here, there is an added wrinkle that the integer must be at least 1000. Therefore, the thousands digit is at least 1. The smallest integer of this type that can be made uses a thousands digit of 1, and then lists the remaining digits in increasing order; this integer is 1023.
The difference between these integers is $3210 - 1023 = 2187$.

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18. The operation $\otimes$ is defined by $a \otimes b = \frac{a}{b} + \frac{b}{a}$. What is the value of $4 \otimes 8$?

(A) $\frac{1}{2}$ (B) 1 (C) $\frac{5}{4}$ (D) 2 (E) $\frac{5}{2}$

Source: 2015 Cayley Grade 10 #11

Primary Topics: Number Sense

Secondary Topics: Operations

Answer: E

Solution:

From the given definition,

$$4 \otimes 8 = \frac{4}{8} + \frac{8}{4} = \frac{1}{2} + 2 = 2\frac{1}{2} = \frac{5}{2}$$

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19. Abigail chooses an integer at random from the set $\{2, 4, 6, 8, 10\}$. Bill chooses an integer at random from the set $\{2, 4, 6, 8, 10\}$. Charlie chooses an integer at random from the set $\{2, 4, 6, 8, 10\}$. What is the probability that the product of their three integers is *not* a power of 2?

(A) $\frac{117}{125}$ (B) $\frac{2}{5}$ (C) $\frac{98}{125}$ (D) $\frac{3}{5}$ (E) $\frac{64}{125}$

Source: 2018 Cayley Grade 10 #20

Primary Topics: Counting and Probability

Secondary Topics: Counting | Divisibility

Answer: C

Solution:

When the product of the three integers is calculated, either the product is a power of 2 or it is not a power of 2.
If p is the probability that the product is a power of 2 and q is the probability that the product is not a power of 2, then $p + q = 1$.
Therefore, we can calculate q by calculating p and noting that $q = 1 - p$.
For the product of the three integers to be a power of 2, it can have no prime factors other than 2. In particular, this means that each of the three integers must be a power of 2.
In each of the three sets, there are 3 powers of 2 (namely, 2, 4 and 8) and 2 integers that are not a power of 2 (namely, 6 and 10).
This means that the probability of choosing a power of 2 at random from each of the sets is $\frac{3}{5}$.
Since Abigail, Bill and Charlie choose their numbers independently, then the probability that each chooses a power of 2 is $\left(\frac{3}{5}\right)^3 = \frac{27}{125}$.
In other words, $p = \frac{27}{125}$ and so $q = 1 - p = 1 - \frac{27}{125} = \frac{98}{125}$.

20. Juliana chooses three different numbers from the set $\{-6, -4, -2, 0, 1, 3, 5, 7\}$ and multiplies them together to obtain the integer n. What is the greatest possible value of n?

(A) 168 (B) 0 (C) 15 (D) 105 (E) 210

Source: 2020 Cayley Grade 10 #14

Primary Topics: Number Sense

Secondary Topics: Optimization | Operations

Answer: A

Solution:

Since $3 \times 5 \times 7 = 105$, then the greatest possible value of n is *at least* 105.

In particular, the greatest possible value of n must be positive.

For the product of three numbers to be positive, either all three numbers are positive (that is, none of the numbers is negative) or one number is positive and two numbers are negative. (If there were an odd number of negative factors, the product would be negative.)

If all three numbers are positive, the product is as large as possible when the three numbers are each as large as possible. In this case, the greatest possible value of n is $3 \times 5 \times 7 = 105$.

If one number is positive and two numbers are negative, their product is as large as possible if the positive number is as large as possible (7) and the product of the two negative numbers is as large as possible.

The product of the two negative numbers will be as large as possible when the negative numbers are each "as negative as possible" (that is, as far from 0 as possible). In this case, these numbers are thus -4 and -6 with product $(-4) \times (-6) = 24$. (We can check the other possible products of two negative numbers and see that none is as large.)

So the greatest possible value of n in this case is $7 \times (-4) \times (-6) = 7 \times 24 = 168$.

Combining the two cases, we see that the greatest possible value of n is 168.

21. If x and y are integers with $(y-1)^{x+y} = 4^3$, then the number of possible values for x is

(A) 8 (B) 3 (C) 4 (D) 5 (E) 6

Source: 2008 Cayley Grade 10 #23

Primary Topics: Algebra and Equations

Secondary Topics: Equations Solving | Exponents

Answer: E

Solution:

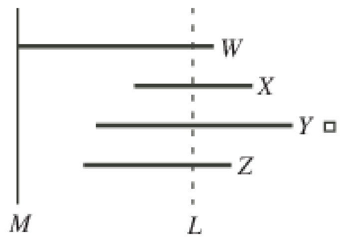
The number 4^3 equals 64.
To express 64 as a^b where a and b are integers, we can use 64^1, 8^2, 4^3, 2^6, $(-2)^6$, and $(-8)^2$. We make a table to evaluate x and y:

$y-1$	$x+y$	y	x
64	1	65	-64
8	2	9	-7
4	3	5	-2
2	6	3	3
-2	6	-1	7
-8	2	-7	9

Therefore, there are 6 possible values for x.

22. Four pieces of lumber are placed in parallel positions, as shown, perpendicular to line M:
    - Piece W is 5 m long
    - Piece X is 3 m long and its left end is 3 m from line M
    - Piece Y is 5 m long and is 2 m from line M
    - Piece Z is 4 m long and is 1.5 m from from line M

    A single cut, perpendicular to the pieces of lumber, is made along the dotted line L. The total length of lumber on each side of L is the same. What is the length, in metres, of the part of piece W to the left of the cut?



(A) 4.25 (B) 3.5 (C) 3.25 (D) 3.75 (E) 4.0

Source: 2008 Pascal Grade 9 #23

Primary Topics: Algebra and Equations

Secondary Topics: Measurement | Decimals

Answer: D

Solution:

Suppose that the distance from line M to line L is d m.
Therefore, the total length of piece W to the left of the cut is d m.
Since piece X is 3 m from line M, then the length of piece X to the left of L is $(d-3)$ m, because 3 of the d m to the left of L are empty.
Similarly, the lengths of pieces Y and Z to the left of line L are $(d-2)$ m and $(d-1.5)$ m.
Therefore, the total length of lumber to the left of line L is

$$d+(d-3)+(d-2)+(d-1.5)=4d-6.5 \text{ m}$$

Since the total length of lumber on each side of the cut is equal, then this total length is $\frac{1}{2}(5+3+5+4)=8.5$ m.
(We could instead find the lengths of lumber to the right of line L to be $5-d$, $6-d$, $7-d$, and $5.5-d$ and equate the sum of these lengths to the sum of the lengths on the left side.)
Therefore, $4d-6.5=8.5$ or $4d=15$ or $d=3.75$, so the length of the part of piece W to the left of L is 3.75 m.

23. A *Fano table* is a table with three columns where
    - each entry is an integer taken from the list $1, 2, 3, \ldots, n$, and
    - each row contains three different integers, and
    - for each possible pair of distinct integers from the list $1, 2, 3, \ldots, n$, there is exactly one row that contains both of these integers.

    The number of rows in the table will depend on the value of n. For example, the table shown is a Fano table with $n = 7$. (Notice that 2 and 6 appear in the same row only once, as does every other possible pair of the numbers $1, 2, 3, 4, 5, 6, 7$.) For how many values of n with $3 \leq n \leq 12$ can a Fano table be created?

1	2	4
2	3	5
3	4	6
4	5	7
5	6	1
6	7	2
7	1	3

(A) 2 (B) 3 (C) 5 (D) 6 (E) 7

Source: 2011 Cayley Grade 10 #23

Primary Topics: Counting and Probability

Secondary Topics: Counting | Operations

Answer: B

Solution:

First, we calculate the number of pairs that can be formed from the integers from 1 to n.
One way to form a pair is to choose one number to be the first item of the pair (n choices) and then a different number to be the second item of the pair ($n-1$ choices).
There are $n(n-1)$ ways to choose these two items in this way.
But this counts each pair twice; for example, we could choose 1 then 3 and we could also choose 3 then 1.
So we have double-counted the pairs, meaning that there are $\frac{1}{2}n(n-1)$ pairs that can be formed.
Next, we examine the number of rows in the table.
Since each row has three entries, then each row includes three pairs (first and second numbers, first and third numbers, second and third numbers).
Suppose that the completed table has r rows.
Then the total number of pairs in the table is $3r$.
Since each pair of the numbers from 1 to n appears exactly once in the table and the total number of pairs from these numbers is $\frac{1}{2}n(n-1)$, then $3r = \frac{1}{2}n(n-1)$, which tells us that $\frac{1}{2}n(n-1)$ must be divisible by 3, since $3r$ is divisible by 3.
We make a table listing the possible values of n and the corresponding values of $\frac{1}{2}n(n-1)$:

n	3	4	5	6	7	8	9	10	11	12
$\frac{1}{2}n(n-1)$	3	6	10	15	21	28	36	45	55	66

Since $\frac{1}{2}n(n-1)$ must be divisible by 3, then the possible values of n are 3, 4, 6, 7, 9, 10, and 12.
Next, consider a fixed number m from the list 1 to n.
In each row that m appears, it will belong to 2 pairs (one with each of the other two numbers in its row).
If the number m appears in s rows, then it will belong to $2s$ pairs.
Therefore, each number m must belong to an even number of pairs.
But each number m from the list of integers from 1 to n must appear in $n-1$ pairs (one with each other number in the list), so $n-1$ must be even, and so n is odd.
Therefore, the possible values of n are 3, 7, 9.
Finally, we must verify that we can create a Fano table for each of these values of n. We are given the Fano table for $n=7$.
Since the total number of pairs when $n=3$ is 3 and when $n=9$ is 36, then a Fano table for $n=3$ will have $3 \div 3 = 1$ row and a Fano table for $n=9$ will have $36 \div 3 = 12$ rows.
For $n=3$ and $n=9$, possible tables are shown below:

1	2	3

1	2	3
1	4	5
1	6	7
1	8	9
2	4	7
2	5	8
2	6	9
3	4	9
3	5	6
3	7	8
4	6	8
5	7	9

In total, there are 3 values of n in this range for which a Fano table can be created.

24. Let n be the largest integer for which $14n$ has exactly 100 digits. Counting from right to left, what is the 68th digit of n?
(A) 1 (B) 2 (C) 4 (D) 5 (E) 8

Source: 2011 Pascal Grade 9 #24

Primary Topics: Number Sense

Secondary Topics: Digits

Answer: A

Solution:

The largest integer with exactly 100 digits is the integer that consists of 100 copies of the digit~9.
This integer is equal to $10^{100} - 1$.
Therefore, we want to determine the largest integer n for which $14n \leq 10^{100} - 1$.
This is the same as trying to determine the largest integer n for which $14n < 10^{100}$, since $14n$ is an integer.
We want to find the largest integer n for which $n < \dfrac{10^{100}}{14} = \dfrac{10}{14} \times 10^{99} = \dfrac{5}{7} \times 10^{99}$.
This is equivalent to calculating the number $\dfrac{5}{7} \times 10^{99}$ and rounding down to the nearest integer.
Put another way, this is the same as calculating $\dfrac{5}{7} \times 10^{99}$ and truncating the number at the decimal point.
The decimal expansion of $\dfrac{5}{7}$ is $0.\overline{714285}$. (We can see this either using a calculator or by doing long division.)
Therefore, the integer that we are looking for is the integer obtained by multiplying $0.\overline{714285}$ by 10^{99} and truncating at the decimal point.
In other words, we are looking for the integer obtained by shifting the decimal point in $0.\overline{714285}$ by 99 places to the right, and then ignoring everything after the new decimal point.
Since the digits in the decimal expansion repeat with period 6, then the integer consists of 16~copies of the digits 714285 followed by 714. (This has $16 \times 6 + 3 = 99$ digits.)
In other words, the integer looks like $714285\ 714285\ \cdots\ 714285\ 714$.
We must determine the digit that is the 68th digit from the right.
If we start listing groups from the right, we first have 714 (3 digits) followed by 11 copies of 714285 (66 more digits). This is 69 digits in total.
Therefore, the "7" that we have arrived at is the 69th digit from the right.
Moving one digit back towards the right tells us that the 68th digit from the right is 1.

25. The number N is the product of all positive odd integers from 1 to 99 that do not end in the digit 5. That is,
$N = 1 \times 3 \times 7 \times 9 \times 11 \times 13 \times 17 \times 19 \times \cdots \times 91 \times 93 \times 97 \times 99$. The units digit of N is

(A) 1 (B) 3 (C) 5 (D) 7 (E) 9

Source: 2012 Gauss Grade 7 #23

Primary Topics: Number Sense

Secondary Topics: Digits | Divisibility

Answer: A

Solution:

The units digit of any product is given by the units digit of the product of the units digits of the numbers being multiplied.
For example, the units digit of the product 12×53 is given by the product 2×3, so it is 6.
Thus to determine the units digit of N, we need only consider the product of the units digits of the numbers being multiplied to give N.
The units digits of the numbers in the product N are $1, 3, 7, 9, 1, 3, 7, 9, \ldots$, and so on.
That is, the units digits $1, 3, 7, 9$ are repeated in each group of four numbers in the product.
There are ten groups of these four numbers, $1, 3, 7, 9$, in the product.
We first determine the units digit of the product $1 \times 3 \times 7 \times 9$.
The units digit of 1×3 is 3.
The units digit of the product 3×7 is 1 (since $3 \times 7 = 21$).
The units digit of 1×9 is 9.
Therefore, the units digit of the product $1 \times 3 \times 7 \times 9$ is 9.
(We could have calculated the product $1 \times 3 \times 7 \times 9 = 189$ to determine the units digit.)
This digit 9 is the units digits of the product of each group of four successive numbers in~N.
Thus, to determine the units digit of N we must determine the units digit of
$9 \times 9 \times 9 \times 9 \times 9 \times 9 \times 9 \times 9 \times 9 \times 9$.
This product is equal to $81 \times 81 \times 81 \times 81 \times 81$.
Since we are multiplying numbers with units digit 1, then the units digit of the product is 1.

26. The list of integers $4, 4, x, y, 13$ has been arranged from least to greatest. How many different possible ordered pairs (x, y) are there so that the average (mean) of these 5 integers is itself an integer?
(A) 7 (B) 8 (C) 9 (D) 10 (E) 11

Source: 2015 Gauss Grade 8 #23

Primary Topics: Counting and Probability

Secondary Topics: Counting | Averages

Answer: E

Solution:

The list of integers $4, 4, x, y, 13$ has been arranged from least to greatest, and so $4 \leq x$ and $x \leq y$ and $y \leq 13$.

The sum of the 5 integers is $4 + 4 + x + y + 13 = 21 + x + y$ and so the average is $\frac{21 + x + y}{5}$.

Since this average is a whole number, then $21 + x + y$ must be divisible by 5 (that is, $21 + x + y$ is a multiple of 5).

How small and how large can the sum $21 + x + y$ be?

We know that $4 \leq x$ and $x \leq y$, so the smallest that $x + y$ can be is $4 + 4 = 8$.

Since $x + y$ is at least 8, then $21 + x + y$ is at least $21 + 8 = 29$.

Using the fact that $x \leq y$ and $y \leq 13$, the largest that $x + y$ can be is $13 + 13 = 26$.

Since $x + y$ is at most 26, then $21 + x + y$ is at most $21 + 26 = 47$.

The multiples of 5 between 29 and 47 are $30, 35, 40$, and 45.

When $21 + x + y = 30$, we get $x + y = 30 - 21 = 9$.

The only ordered pair (x, y) such that $4 \leq x$ and $x \leq y$ and $y \leq 13$, and $x + y = 9$ is $(x, y) = (4, 5)$. Continuing in this way, we determine all possible values of x and y that satisfy the given conditions in the table below.

Value of $21 + x + y$	**Value of** $x + y$	**Ordered Pairs** (x, y) **with** $4 \leq x$ **and** $x \leq y$ **and** $y \leq 13$
30	$30 - 21 = 9$	$(4, 5)$
35	$35 - 21 = 14$	$(4, 10), (5, 9), (6, 8), (7, 7)$
40	$40 - 21 = 19$	$(6, 13), (7, 12), (8, 11), (9, 10)$
45	$45 - 21 = 24$	$(11, 13), (12, 12)$

The number of ordered pairs (x, y) such that the average of the 5 integers $4, 4, x, y, 13$ is itself an integer is 11.

27. There are n students in the math club at Scoins Secondary School. When Mrs. Fryer tries to put the n students in groups of 4, there is one group with fewer than 4 students, but all of the other groups are complete. When she tries to put the n students in groups of 3, there are 3 more complete groups than there were with groups of 4, and there is again exactly one group that is not complete. When she tries to put the n students in groups of 2, there are 5 more complete groups than there were with groups of 3, and there is again exactly one group that is not complete. The sum of the digits of the integer equal to $n^2 - n$ is
(A) 11 (B) 12 (C) 20 (D) 13 (E) 10

Source: 2016 Pascal Grade 9 #22

Primary Topics: Number Sense

Secondary Topics: Divisibility

Answer: B

Solution:

Solution 1
Suppose that, when the n students are put in groups of 2, there are g complete groups and 1 incomplete group.
Since the students are being put in groups of 2, an incomplete group must have exactly 1 student in it.
Therefore, $n = 2g + 1$.
Since the number of complete groups of 2 is 5 more than the number of complete groups of 3, then there were $g - 5$ complete groups of 3.
Since there was still an incomplete group, this incomplete group must have had exactly 1 or 2 students in it.
Therefore, $n = 3(g - 5) + 1$ or $n = 3(g - 5) + 2$.
If $n = 2g + 1$ and $n = 3(g - 5) + 1$, then $2g + 1 = 3(g - 5) + 1$ or $2g + 1 = 3g - 14$ and so $g = 15$.
In this case, $n = 2g + 1 = 31$ and there were 15 complete groups of 2 and 10 complete groups of 3.
If $n = 2g + 1$ and $n = 3(g - 5) + 2$, then $2g + 1 = 3(g - 5) + 2$ or $2g + 1 = 3g - 13$ and so $g = 14$.
In this case, $n = 2g + 1 = 29$ and there were 14 complete groups of 2 and 9 complete groups of 3.
If $n = 31$, dividing the students into groups of 4 would give 7 complete groups of 4 and 1 incomplete group.
If $n = 29$, dividing the students into groups of 4 would give 7 complete groups of 4 and 1 incomplete group.
Since the difference between the number of complete groups of 3 and the number of complete groups of 4 is given to be 3, then it must be the case that $n = 31$.
In this case, $n^2 - n = 31^2 - 31 = 930$; the sum of the digits of $n^2 - n$ is 12.

Solution 2
Since the n students cannot be divided exactly into groups of 2, 3 or 4, then n is not a multiple of 2, 3 or 4.
The first few integers larger than 1 that are not divisible by 2, 3 or 4 are 5, 7, 11, 13, 17, 19, 23, 25, 29, 31, and 35.
In each case, we determine the number of complete groups of each size:

n	5	7	11	13	17	19	23	25	29	31	35
# of complete groups of 2	2	3	5	6	8	9	11	12	14	15	17
# of complete groups of 3	1	2	3	4	5	6	7	8	9	10	11
# of complete groups of 4	1	1	2	3	4	4	5	6	7	7	8

Since the number of complete groups of 2 is 5 more than the number of complete groups of 3 which is 3 more than the number of complete groups of 4, then of these possibilities, $n = 31$ works.
In this case, $n^2 - n = 31^2 - 31 = 930$; the sum of the digits of $n^2 - n$ is 12.
(Since the problem is a multiple choice problem and we have found a value of n that satisfies the given conditions and for which an answer is present, then this answer must be correct. Solution 1 shows why $n = 31$ is the only value of n that satisfies the given conditions.)

28. The number 385 is an example of a three-digit number for which one of the digits is the sum of the other two digits. How many numbers between 100 and 999 have this property?

(A) 144 (B) 126 (C) 108 (D) 234 (E) 64

Source: 2022 Gauss Grade 8 #24

Primary Topics: Number Sense

Secondary Topics: Counting | Operations | Digits

Answer: B

Solution:

Solution 1:

We begin by recognizing that numbers with the given property cannot have two digits that are zero.
Can you see why?
Thus, numbers with this property have exactly one zero or they have no zeros.
We consider each of these two cases separately.

Case 1: Suppose the number has exactly one digit that is a zero.

Each of the numbers greater than 100 and less than 999 is a three-digit number and so in this case, the number also has two non-zero digits.
Since one of the digits is equal to the sum of the other two digits and one of the digits is zero, then the two non-zero digits must be equal to one another.
The two non-zero digits can equal any integer from 1 to 9, and thus there are 9 possible values for the non-zero digits.
For each of these 9 possibilities, the zero digit can be the second digit in the number or it can be the third digit (the first digit cannot be zero).
That is, there are 9 possible values for the non-zero digits and 2 ways to arrange the three digits, and thus $9 \times 2 = 18$ numbers of this form satisfy the given property.
For example, numbers of this form are 101 and 110, 202 and 220, and so on.

Case 2: Suppose the number has no digits that are equal to zero.

Let the three digits of the number be a, b and c, arranged in some order.
Assume that a is the largest digit, and so $a = b + c$.
If $a = b$, then $c = 0$ which contradicts our assumption that no digit is equal to 0.
Similarly, if $a = c$, then $b = 0$ and the same contradiction arises.
Thus $a > b$ and $a > c$.
If $a = 1$, then $b = c = 0$ (since $a > b$ and $a > c$), but then $a \neq b + c$.
Therefore, a is at least 2.
If $a = 2$, then $b = c = 1$ and these are the only possible values for b and c when $a = 2$.
In this case, the 3 ways to arrange these digits give the numbers 112, 121 and 211, each with the desired property.
If $a = 3$, then b and c equal 1 and 2, in some order.
In this case, the 6 ways to arrange these digits give the numbers 123, 132, 213, 231, 312, and 321, each with the desired property.
From these cases, we recognize that if $b = c$, then there are 3 possible arrangements of the digits.
However, if the three digits are different from one another, then there are 3 choices for the first digit, 2 choices for second digit and 1 choice for the third digit, and thus $3 \times 2 \times 1 = 6$ ways to arrange the digits.
We consider all possible values of a, b, c and count the arrangements of these digits in the table below.

Values for a	Values for b and c with the number of arrangements in brackets [ ]	Total number of arrangements
2	1,1 [3]	3
3	$1, 2$ [6]	6
4	$1, 3$ [6]; $2, 2$ [3]	9
5	$1, 4$ [6]; $2, 3$ [6]	12
6	$1, 5$ [6]; $2, 4$ [6]; $3, 3$ [3]	15
7	$1, 6$ [6]; $2, 5$ [6]; $3, 4$ [6]	18
8	$1, 7$ [6]; $2, 6$ [6]; $3, 5$ [6]; $4, 4$ [3]	21
9	$1, 8$ [6]; $2, 7$ [6]; $3, 6$ [6]; $4, 5$ [6]	24

Thus, there are $18 + 3 + 6 + 9 + 12 + 15 + 18 + 21 + 24 = 126$ numbers that satisfy the given property.

Solution 2:

We begin by recognizing that numbers with the given property cannot have three equal digits. Can you see why?

Thus, numbers with this property have exactly two digits that are equal or all three digits are different.
We consider each of these two cases separately.

Case 1: Suppose the number has exactly two digits that are equal.

The two equal digits cannot be 0 since then the third digit would also be 0.
If for example the two equal digits are 1s, there are two possibilities for the third digit.
The third digit can be 2 (since $1 + 1 = 2$) or the third digit can be 0 (since $1 + 0 = 1$).
In the cases for which the third digit is equal to the sum of the two equal digits, the equal digits can be $1, 2, 3$, or 4 and the third digit is $2, 4, 6, 8$, respectively.
(We note that the equal digits cannot be greater than 4 since their sum is greater than 9.)
For each of these 4 possibilities, there are 3 ways to arrange the digits.
For example, when the equal digits are 1s and the third digit is 2, the numbers 112, 121 and 211 have the desired property.
Thus, there are $4 \times 3 = 12$ such numbers for which the third digit is equal to the sum of the two equal digits.
In the cases for which two of the digits are equal and the third digit is 0, the equal digits can be any integer from 1 to 9 inclusive.
For each of these 9 possibilities, there are 2 ways to arrange the digits.
For example, when the equal digits are 1s and the third digit is 0, the numbers 101 and 110 have the desired property.
Thus, there are $9 \times 2 = 18$ numbers which have two equal digits and the third digit is 0.
In total, there are $12 + 18 = 30$ numbers which have two equal digits and satisfy the given property.

Case 2: Suppose all three digits are different from one another.

Let the three digits of the number be a, b and c, arranged in some order with $a > b > c$.
Since a is the largest digit, then $a = b + c$.
If $a = 1$, then $b = c = 0$ (since $a > b$ and $a > c$), but this is not possible since $b > c$.
Similarly, if $a = 2$, then $b = 1$ and $c = 0$, however these digits do not satisfy the given property.
Therefore, a is at least 3.
If $a = 3$, then $b = 2$ and $c = 1$.
In this case, the 6 ways to arrange these digits give the numbers 123, 132, 213, 231, 312, 321 and each has the desired property.
We consider all possible values of a, b, c in the table below.

Values for a	**Values for b, c**
3	$2, 1$
4	$3, 1$
5	$4, 1$; $3, 2$
6	$5, 1$; $4, 2$
7	$6, 1$; $5, 2$; $4, 3$
8	$7, 1$; $6, 2$; $5, 3$
9	$8, 1$; $7, 2$; $6, 3$; $5, 4$

For each of these 16 possibilities in the table above, there are 6 ways to arrange the three digits, and so there are $16 \times 6 = 96$ such numbers.
Thus, there are a total of $30 + 96 = 126$ numbers that satisfy the given property.

29. It took Nasrin two hours and thirty minutes to canoe the 4.5 km into her camp. Paddling much faster, the return trip took her $\frac{1}{3}$ of the time. What was Nasrin's mean (average) speed as she paddled to camp and back?
(A) 1.25 km/h (B) 3.96 km/h (C) 1.8 km/h (D) 1.95 km/h (E) 2.7 km/h

Source: 2023 Gauss Grade 8 #22

Primary Topics: Algebra and Equations

Secondary Topics: Rates | Expressions | Equations Solving | Averages

Answer: E

Solution:

Nasrin's mean (average) speed is determined by dividing the total distance travelled, which is 9 km, by the total time. It took Nasrin 2 hours and thirty minutes, or 150 minutes, to canoe into her camp. On the return trip, it took her $\frac{1}{3} \times 150$ minutes or 50 minutes.
Thus, the total time for Nasrin to paddle to camp and back was 200 minutes. Converting to hours, 200 minutes is 3 hours and 20 minutes, and since 20 minutes is $\frac{20}{60} = \frac{1}{3}$ hours, it took Nasrin $3\frac{1}{3}$ hours in total. Thus, Nasrin's mean speed as she paddled to camp and back was $\frac{9\text{ km}}{3\frac{1}{3}\text{ h}}$ or $\frac{9\text{ km}}{\frac{10}{3}\text{ h}}$, which is equal to $9 \times \frac{3}{10}\text{ km/h} = \frac{27}{10}\text{ km/h} = 2.7\text{ km/h}$.

30. A *Pretti number* is a seven-digit positive integer with the following properties:

    - The integer formed by its leftmost three digits is a perfect square.
    - The integer formed by its rightmost four digits is a perfect cube.
    - Its ten thousands digit and ones (units) digit are equal.
    - Its thousands digit is not zero.

    How many Pretti numbers are there?

Source: 2022 Pascal Grade 9 #24

Primary Topics: Other | Number Sense

Secondary Topics: Counting | Logic | Digits

Answer: 30

Solution:

Since a Pretti number has 7 digits, it is of the form $a\,bcd\,efg$.
From the given information, the integer with digits abc is a perfect square.
Since a Pretti number is a seven-digit positive integer, then $a > 0$, which means that abc is between 100 and 999, inclusive.
Since $9^2 = 81$ (which has two digits) $10^2 = 100$ (which has three digits) and $31^2 = 961$ (which has three digits) and $32^2 = 1024$ (which has four digits), then abc (which has three digits) must be one of $10^2, 11^2, \ldots, 30^2, 31^2$, since 32^2 has 4 digits..
From the given information, the integer with digits $defg$ is a perfect cube.
Since the thousands digit of a Pretti number is not 0, then $d > 0$.
Since $9^3 = 729$ and $10^3 = 1000$ and $21^3 = 9261$ and $22^3 = 10\,648$, then $defg$ (which has four digits) must be one of $10^3, 11^3, \ldots, 20^3, 21^3$, since 22^3 has 5 digits.
Since the ten thousands digit and units digit of the original number are equal, then $c = g$.
In other words, the units digits of abc and $defg$ are equal.
The units digit of a perfect square depends only on the units digit of the integer being squared, since in the process of multiplication no digit to the left of this digit affects the resulting units digit.
The squares 0^2 through 9^2 are $0, 1, 4, 9, 16, 25, 36, 49, 64, 81$.
This gives the following table:

Units digit of n^2	**Possible units digits of n**
0	0
1	1, 9
4	2, 8
5	5
6	4, 6
9	3, 7

Similarly, the units digit of a perfect cube depends only on the units digit of the integer being cubed.
The cubes 0^3 through 9^3 are $0, 1, 8, 27, 64, 125, 216, 343, 512, 729$.
This gives the following table:

Units digit of m^3	**Possible units digits of m**
0	0
1	1
2	8
3	7
4	4
5	5
6	6
7	3
8	2
9	9

We combine this information to list the possible values of $c = g$ (from the first table, these must be 0, 1, 4, 5, 6, 9), the squares between 10^2 and 31^2, inclusive, with this units digit, and the cubes between 10^3 and 21^3 with this units digit:

Digit $c = g$	**Possible squares**	**Possible cubes**	**Pretti numbers**
0	10^2, 20^2, 30^2	10^3, 20^3	$3 \times 2 = 6$
1	11^2, 19^2, 21^2, 29^2, 31^2	11^3, 21^3	$5 \times 2 = 10$
4	12^2, 18^2, 22^2, 28^2	14^3	$4 \times 1 = 4$
5	15^2, 25^2	15^3	$2 \times 1 = 2$
6	14^2, 16^2, 24^2, 26^2	16^3	$4 \times 1 = 4$
9	13^2, 17^2, 23^2, 27^2	19^3	$4 \times 1 = 4$

For each square in the second column, each cube in the third column of the same row is possible. (For example, 19^2 and 11^3 give the Pretti number $3\,611\,331$ while 19^2 and 21^3 give the Pretti number $3\,619\,261$.) In each case, the number of Pretti numbers is thus the product of the number of possible squares and the number of possible cubes.
Therefore, the number of Pretti numbers is $6+10+4+2+4+4=30$.